Spatio-temporal constrained zonotopes for validation of optimal control problems

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Motivation

Robust control!

Motivation

Guiding a launch vehicle =
Optimal Control Problem (OCP)

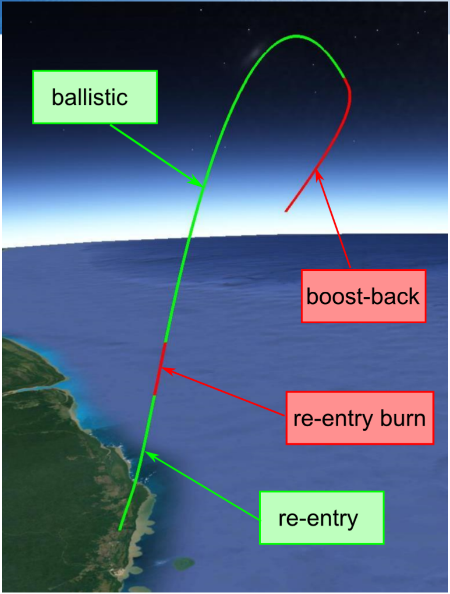
OCP formulation

$$\min_{u(\cdot)} \int_0^{t_f} \ell(y(t), u(t), \xi) dt$$

$$\text{s.t.} \begin{cases} \dot{y}(t) = f(y(t), u(t), \xi), \\ y(0) = y_0, \\ y(t_f) \in \mathcal{Y}_f, \\ t_f \text{ is free.} \end{cases}$$

Model not exact!
Depends on

- parameters ξ,
- initial state y_0.



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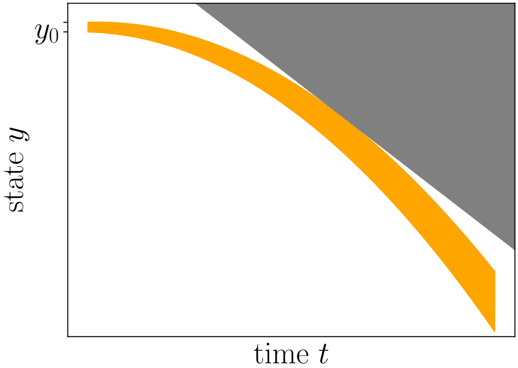
Motivation

Hypothesis:
bounded uncertainties on parameters and initial state.
$\xi \in [\xi]$ and $y_0 \in [y_0]$

Dynamics with uncertainties

$$\begin{cases} \dot{y} \in [f](y, u, [\xi]) \\ y(0) \in [y_0] \end{cases}$$

Goal: enclose optimal trajectories, assess risks



Orange: Possible trajectories of a falling ball with uncertainties
Grey: unsafe set

Validated methods

Principle:

1. enclose results in sets : $[\pi] = [3.14, 3.15]$
2. replace function f with set valued $[f]$ s.t. $[f]([a]) \supseteq \{f(a) | \forall a \in [a]\}$.

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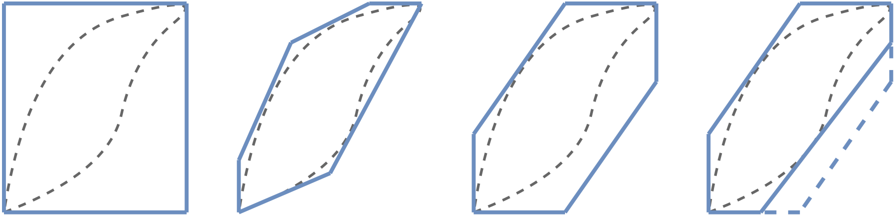

Interval vector — Polytope — Zonotope — Constrained zonotope

Interval vector: Cheap but significant over approximation

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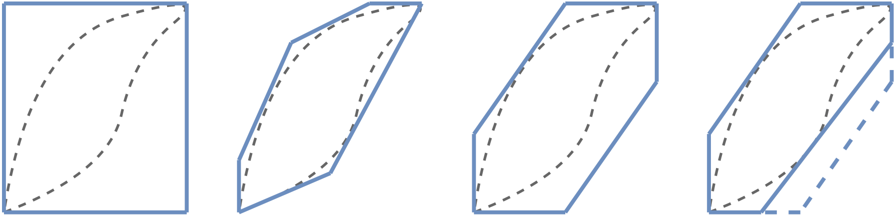

Interval vector — Cheap but significant over approximation

Polytope — Precise but expensive

Zonotope

Constrained zonotope

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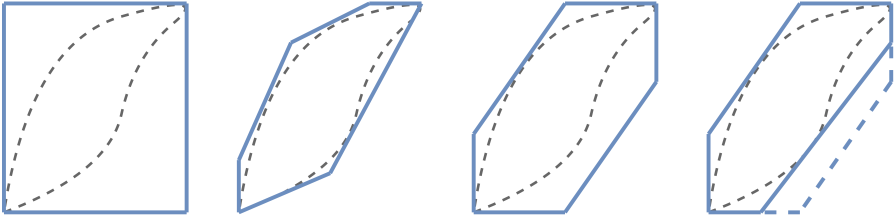


Interval vector	Polytope	Zonotope	Constrained zonotope
Cheap but significant over approximation	Precise but expensive	Good middle ground	Good middle ground

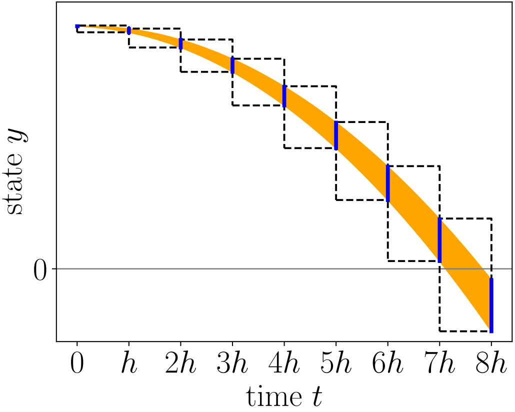
Validated simulation

Let an uncontrolled system:

$$\begin{cases} \dot{x} \in [g](x, [\xi]) \\ x(0) \in [x_0] \end{cases}$$

Validated simulation = enclosure in a sequence of boxes (dashed) and zonotopes (blue).

DynIbex = C++ library with validated Runge Kutta methods and zonotopes.



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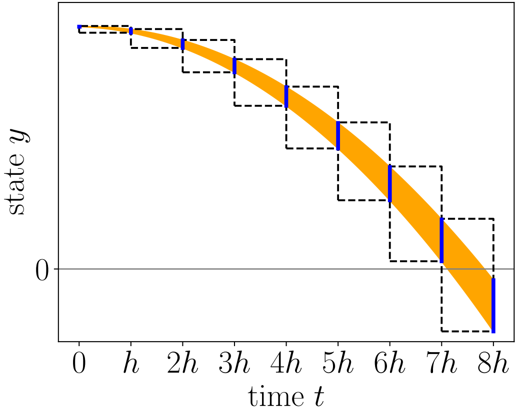

What if trajectories are subject to a control defined implicitely as solution of an OCP?

Characterization of optimal trajectories

Pontryagin's Maximum Principle (PMP)

Optimal trajectories are caracterized by an uncontrolled switched system with constraints:

$$\begin{cases} \dot{x}(t) = g_n(x(t), \xi), & \forall t \in [\Theta_{n-1}^+, \Theta_n^-] \\ x(0) = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}, \end{cases}$$

with constraints

$$C_n(x(\Theta_n^-)) = 0, \forall n \in 1..N,$$

Variables :

- initial co-state $p_0 \in \mathbb{R}^n$,
- transition times $0 < \Theta_1 < \ldots < \Theta_N = t_f$.

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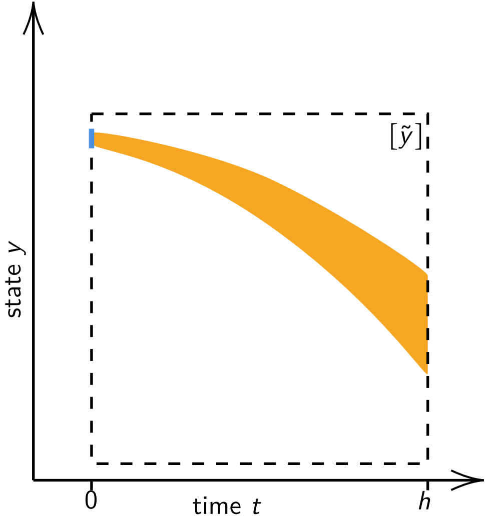
Problem : deal with variable transition times

Building spatio temporal zonotopes with validated Taylor

$[\tilde{y}]$= Picard box that encloses all trajectories over time range $[0, h]$



Building spatio temporal zonotopes with validated Taylor

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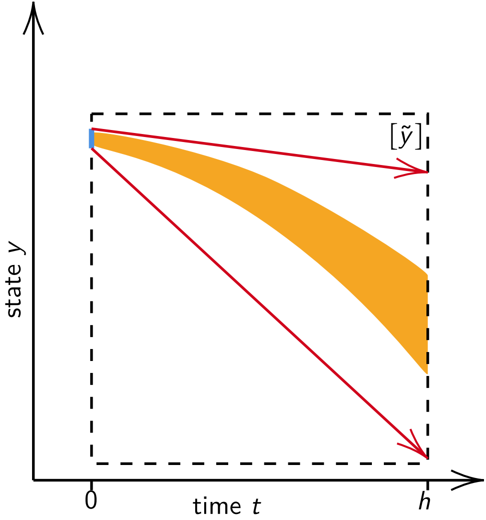


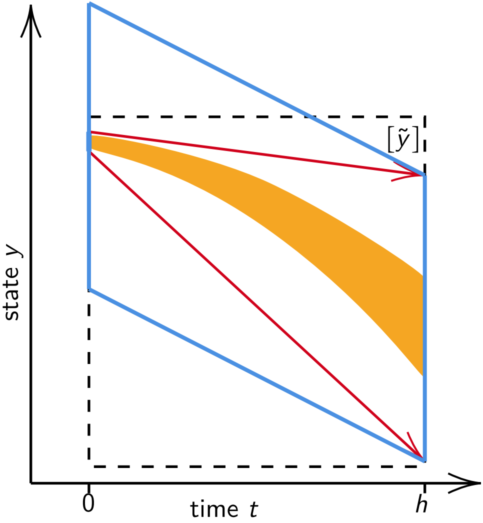
Building spatio temporal zonotopes with validated Taylor

$[\tilde{y}]$= Picard box that encloses all trajectories over time range $[0, h]$

Subsequent derivatives of f over $[\tilde{y}]$ are enclosed

A validated Taylor interpolation yields a zonotope enclosing trajectories over time range $[0, h]$

Spatio temporal zonotopes = state + time coordinates

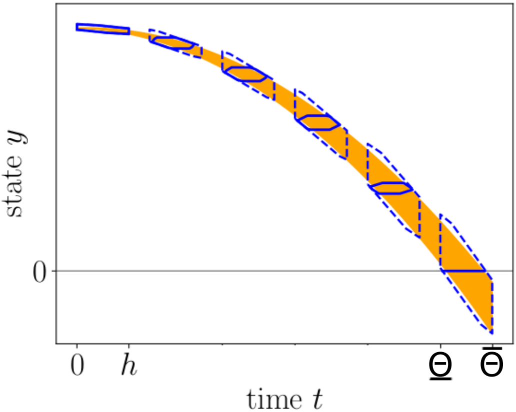


Constrained spatio temporal zonotopes

Let a variable transition time $\Theta \in [\underline{\Theta}, \overline{\Theta}]$.

1. take $h = \overline{\Theta} - \underline{\Theta}$,
2. enclose trajectories over $[\underline{\Theta}, \overline{\Theta}]$ in a zonotope,
3. add $C(x(\Theta^-)) = 0$ as constraints,
4. propagate constraints backward with guaranteed linearization.

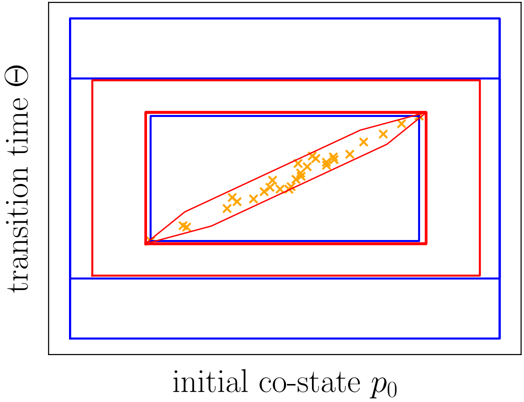
Problem : how do we know bounds $\underline{\Theta}$ and $\overline{\Theta}$?



Dashed: spatio temporal zonotopes
Plain: zonotopes + optimality condition

Enclosing variables with an inflate & contract method

Problem : need an enclosure of the variables.



Inflate & contract method:

Start with a box enclosing numerical solutions, inflate it until it contains all solutions.

Contract the box with fixed point iterations.

→ validated enclosure of all variables

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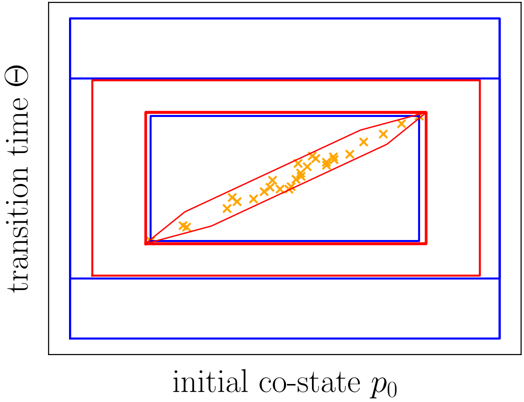

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Start with a box enclosing numerical solutions, inflate it until it contains all solutions.

Contract the box with fixed point iterations.

→ validated enclosure of all variables

→ self contained method

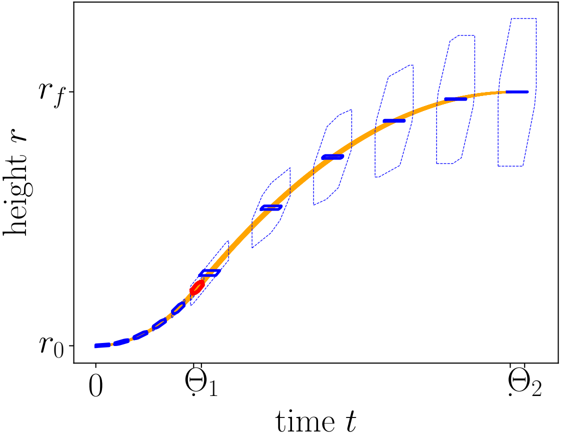
Back to aerospace problems

Consider a simple take-off problem:

Goddard's problem

$$\min_{u(\cdot)} \int_0^{t_f} |u| dt$$

$$\text{s.t.} \begin{cases} \dot{r}(t) = v, \\ \dot{v}(t) = -\frac{G}{r^2} + \frac{Cu}{m}, \\ \dot{m}(t) = -b|u|, \\ y(0) = y_0 \\ r(t_f) = r_f, \\ v(t_f), m(t_f), t_f \\ \text{ are free.} \end{cases}$$



Orange: trajectories for various values of the parameters. They are enclosed as intended.

Conclusion

Our method:

1. OCP $\rightarrow$ uncontrolled switched system,
2. enclose system at transition time with spatio temporal zonotopes,
3. add optimality conditions as constraints, propagate them backward,
4. inflate & contract method.

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Future works:

- more complex aerospace problems,
- decrease the over approximation.

Thank you for your attention

- J. Alexandre dit Sandretto and A. Chapoutot.
  Validated explicit and implicit Runge–Kutta methods.
  Reliable Computing, 22(1):79–103, Jul 2016.
- Etienne Bertin, Elliot Brendel, Bruno Hérissé, Julien Alexandre dit Sandretto, and Alexandre Chapoutot.
  Prospects on solving an optimal control problem with bounded uncertainties on parameters using interval arithmetics.
  Acta Cybernetica, Feb. 2021.
- F. Bonnans, P. Martinon, and E. Trélat.
  Singular arcs in the generalized goddard's problem.
  Journal of Optimization Theory and Applications, 139(2):439–461, 2008.
- Joseph K. Scott, Davide M. Raimondo, Giuseppe Roberto Marseglia, and Richard D. Braatz.
  Constrained zonotopes: A new tool for set-based estimation and fault detection.
  Automatica, 69:126 – 136, 2016.